Proc. Natl. Acad. Sci. USA
Vol. 89, pp. 8177–8179, September 1992
Mathematics

Canonical bases in tensor products

(quantized enveloping algebra/R-matrix/highest weight representation/coordinate algebra)

G. Lusztig

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

Communicated by Bertram Kostant, April 17, 1992

ABSTRACT **I construct a canonical basis in the tensor product of a simple integrable highest weight module with a simple integrable lowest weight module of a quantized enveloping algebra. This basis is simultaneously compatible with many submodules of the tensor product. As an application, I obtain a construction of a canonical basis of (a modified form of) the quantized enveloping algebra.**

Section 1: Notations. Let Y be a free abelian group of finite type and let $X = \mathrm{Hom}(Y, \mathbf{Z})$. I assume that we are given linearly independent subsets $I \subset Y$, $I' \subset X$ in bijection $i \leftrightarrow i'$ such that $(i'(j))_{i,j\in I}$ is a generalized Cartan matrix, which for simplicity is assumed to be symmetric (although the results hold without this assumption). Let $\mathbf{U}$ be the Hopf algebra over $\mathbf{Q}(v)$ (v is an indeterminate) attached by Drinfeld (1) and Jimbo to these data; this is a quantum version of the universal enveloping algebra of the Lie algebra over $\mathbf{Q}$ attached by Kac and Moody to the same data. The standard generators are E_i, F_i ($i \in I$) and K_y ($y \in Y$); the relations are $K_yE_i = v^{i'(y)}E_iK_y$; $K_yF_i = v^{-i'(y)}F_iK_y$; $E_iF_j - F_jE_i = \delta_{ij}(K_i - K_{-i})/(v - v^{-1})$; $K_yK_{y'} = K_{y+y'}$; and the v-analogs of the Serre relations. The comultiplication is given by $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$; $\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_{-i}$; $\Delta(K_y) = K_y \otimes K_y$. Let $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) be the subalgebra of $\mathbf{U}$ generated by the E_i (resp. by the F_i). For any $\nu \in \mathbf{N}^I$, let $\mathbf{U}^+_\nu$ (resp. $\mathbf{U}^-_\nu$) be the subspace of $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) spanned by words in the E_i (resp. F_i) in which E_j (resp. F_j) occurs $\nu(j)$ times for each j. Then $\mathbf{U}^+ = \oplus_\nu \mathbf{U}^+_\nu$, $\mathbf{U}^- = \oplus_\nu \mathbf{U}^-_\nu$. Let $X^+ = \{x \in X | x(i) \in \mathbf{N} \text{ for all } i\}$. For any $x \in X^+$, let (V_x, ξ_x) be a simple (integrable) $\mathbf{U}$-module with a given generating vector ξ_x such that $F_i\xi_x = 0$, $K_y\xi_x = v^{-x(y)}\xi_x$ for all i, y; let (Λ_x, η_x) be a simple (integrable) $\mathbf{U}$-module with a given generating vector η_x such that $E_i\eta_x = 0$, $K_y\eta_x = v^{x(y)}\eta_x$ for all i, y. A canonical basis B^+ of $\mathbf{U}^+$ with very favorable properties is described in ref. 2 for types ADE, and a general definition has been given in refs. 3 and 4. Here I shall use the definition in ref. 3. Let B^- be the analogous basis of $\mathbf{U}^-$. For any $x \in X^+$, let $B^+_x = \{b \in B^+ | b\xi_x \neq 0\}$ and $B^-_x = \{b \in B^- | b\eta_x \neq 0\}$. Then $b \mapsto b\xi_x$ (resp. $b \mapsto b\eta_x$) defines a bijection of B^+_x (resp. B^-_x) onto a basis $\mathcal{B}^+_x$ of V_x (resp. onto a basis $\mathcal{B}^-_x$ of Λ_x); these are the canonical bases of V_x, Λ_x. Let $A = \mathbf{Z}[v, v^{-1}]$. Then $\mathbf{U}$ has a natural A-form $\mathbf{U}_A$ (with divided powers). Moreover, V_x, Λ_x have natural A-lattices $V_{x,A}$, $\Lambda_{x,A}$, generated by $\mathcal{B}^+_x$, $\mathcal{B}^-_x$, respectively; these lattices are U_A-stable. Let $\mathcal{L}_x$ (resp. $\mathcal{L}'_x$) be the $\mathbf{Z}[v^{-1}]$-submodule of V_x (resp. Λ_x) generated by $\mathcal{B}^+_x$ (resp. $\mathcal{B}^-_x$).

Section 2. The main result of this paper is the construction of a canonical basis of a tensor product $V_x \otimes \Lambda_z$ (where $x, z \in X^+$). This basis has a remarkable stability property that makes it simultaneously compatible with many natural subspaces of the tensor product. As an application, I construct a canonical basis of a (modified form of) $\mathbf{U}$, in which the structure constants are in A and are (conjecturally) in $\mathbf{N}[v, v^{-1}]$.

The publication costs of this article were defrayed in part by page charge payment. This article must therefore be hereby marked "*advertisement*" in accordance with 18 U.S.C. §1734 solely to indicate this fact.

Section 3: The quasi-R-matrix. Let $^-: \mathbf{U} \to \mathbf{U}$ be the (involutive) $\mathbf{Q}$-algebra homomorphism defined by

$$\overline{E}_i = E_i,\ \overline{F}_i = F_i,\ \overline{K}_y = K_{-y},\ \overline{v} = v^{-1}.$$

Let $^-: \mathbf{U} \otimes \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ be the $\mathbf{Q}$-algebra homomorphism defined by $^- \otimes {}^-$. Let $\overline{\Delta}: \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ be the $\mathbf{Q}(v)$-algebra homomorphism defined by $\overline{\Delta}(u) = \overline{\Delta(\overline{u})}$ for all $u \in \mathbf{U}$.

PROPOSITION 1. *There exist uniquely defined elements $\Theta_\nu \in \mathbf{U}^-_\nu \otimes \mathbf{U}^+_\nu$ (for $\nu \in \mathbf{N}^I$) such that $\Theta_0 = 1 \otimes 1$ and $\Theta = \Sigma_\nu \Theta_\nu$ satisfies $\Delta(u)\Theta = \Theta\overline{\Delta}(u)$ for all $u \in \mathbf{U}$ (equality in a suitable completion of $\mathbf{U} \otimes \mathbf{U}$). We have $\Theta\overline{\Theta} = \overline{\Theta}\Theta = 1 \otimes 1$.* The existence of Θ is similar to the existence (1, 5) of Drinfeld's universal R-matrix of $\mathbf{U}$. In fact, one shows that Θ is obtained from Drinfeld's element by removing the Cartan part and by transposing the factors. (For this reason, Θ is called the *quasi-R-matrix.*) The uniqueness of Θ is easier than that of the Drinfeld element (which requires additional properties). The last assertion of the proposition follows from uniqueness.

For example, in type A_1, with $I = \{i\}$ (with notation of ref. 2),

$$\Theta = \sum_{k\geq 0} (-1)^k v^{-k(k-1)/2}(v - v^{-1})(v^2 - v^{-2}) \cdots (v^k - v^{-k})F_i^{(k)} \otimes E_i^{(k)}.$$

Section 4. Let $x, z \in X^+$ and regard $V_x \otimes \Lambda_z$ as a $\mathbf{U}$-module, via Δ. We denote by $\alpha_{x,z}: \mathbf{U} \to V_x \otimes \Lambda_z$ the (surjective) linear map given by $u \mapsto u(\xi_x \otimes \eta_z)$.

Let $^-: V_x \to V_x$ (resp. $^-: \Lambda_z \to \Lambda_z$) be the unique $\mathbf{Q}$-linear map such that $\overline{u\xi_x} = \overline{u}\xi_x$ (resp. $\overline{u\eta_z} = \overline{u}\eta_z$) for all $u \in \mathbf{U}$. Let $^-: V_x \otimes \Lambda_z \to V_x \otimes \Lambda_z$ be defined by $^- \otimes {}^-$. We can regard Θ as a $\mathbf{Q}(v)$-linear endomorphism of $V_x \otimes \Lambda_z$ (using the $\mathbf{U} \otimes \mathbf{U}$-module structure); any given vector is annihilated by all but finitely many Θ_ν. Let $\Psi: V_x \otimes \Lambda_z \to V_x \otimes \Lambda_z$ be the $\mathbf{Q}$-linear map given by $\Psi(r) = \Theta(\overline{r})$. I state some properties of Ψ.

(*i*) $\alpha_{x,z}(\overline{u}) = \Psi(\alpha_{x,z}(u))$ for all $u \in \mathbf{U}$. This follows from the definition of Θ and the fact that $\xi_x \otimes \eta_z$ is fixed both by Θ and $^-$.

(*ii*) Ψ maps the A-lattice $V_{x,A} \otimes_A \Lambda_{z,A}$ into itself.

Indeed, any element in that lattice is of the form $\alpha_{x,z}(u)$ for some $u \in \mathbf{U}_A$. Its image under Ψ is $\alpha_{x,z}(\overline{u})$; this is again in the lattice since $\mathbf{U}_A$ is stable under $^-$.

If $b \in B^+$ (resp. $b \in B^-$), we set $|b| = \Sigma_i\nu(i)$ where $\nu \in \mathbf{N}^I$ is such that $b \in \mathbf{U}^+_\nu$ (resp. $b \in \mathbf{U}^-_\nu$).

If $b \in B^+_x$ and $b' \in B^-_z$, then $b\xi_x \otimes b'\eta_z$ is fixed by $^-$; using this and the general form of Θ, we see the following.

(*iii*) $\Psi(b\xi_x \otimes b'\eta_z) = b\xi_x \otimes b'\eta_z + \Sigma_{b_1,b_1'}c(b, b', b_1, b_1')b_1\xi_x \otimes b_1'\eta_z$ sum over all $b_1 \in B^+_x$ and $b_1' \in B^-_z$ such that $|b_1| < |b|$ and $|b_1'| < |b'|$; the coefficients $c(b, b', b_1, b_1')$ are in $\mathbf{Q}(v)$. Combining properties *ii* and *iii* we obtain the following.

(*iv*) The coefficients $c(b, b', b_1, b_1')$ in property *iii* are in A. From the definitions and from the last assertion of *Proposition 1* we deduce property *v*.

(*v*) $\Psi^2 = 1$ and Ψ is antilinear with respect to $v \to v^{-1}$.

Let $\mathcal{L}_{x,z} = \mathcal{L}_x \otimes_{\mathbf{Z}[v^{-1}]} \mathcal{L}'_z$ be the $\mathbf{Z}[v^{-1}]$-submodule of $V_x \otimes \Lambda_z$ generated by the basis $\mathcal{B}^+_x \otimes \mathcal{B}^-_z$.

THEOREM 1. (i) *The natural map*

$$\pi : \mathcal{L}_{x,z} \cap \Psi(\mathcal{L}_{x,z}) \to \mathcal{L}_{x,z}/v^{-1}\mathcal{L}_{x,z}$$

is an isomorphism of abelian groups.

(ii) *For any* $(b, b') \in B_x^+ \times B_z^-$, *there is a unique element* $(b \diamond b')_{x,z} \in \mathcal{L}_{x,z}$ *such that* $\Psi((b \diamond b')_{x,z}) = (b \diamond b')_{x,z}$ *and* $\pi((b \diamond b')_{x,z}) = \pi(b\xi_x \otimes b'\eta_z)$.

(iii) *The element* $(b \diamond b')_{x,z}$ *in assertion ii is equal to* $b\xi_x \otimes b'\eta_z$ *plus a* $\mathbf{Z}[v^{-1}]$*-linear combination of elements* $b_1\xi_x \otimes b_1'\eta_z$, *with* $b_1 \in B_x^+$ *and* $b_1' \in B_z^-$ *such that* $|b_1| < |b|$ *and* $|b_1'| < |b'|$.

(iv) *The elements* $(b \diamond b')_{x,z}$ *with* b, b′ *as above form a* $\mathbf{Q}(v)$*-basis of* $V_x \otimes \Lambda_z$, *an* A*-basis of* $V_{x,A} \otimes_A \Lambda_{z,A}$, *a* $\mathbf{Z}[v^{-1}]$*-basis of* $\mathcal{L}_{x,z}$ *and a* $\mathbf{Z}$*-basis of* $\mathcal{L}_{x,z}/v^{-1}\mathcal{L}_{x,z}$.

This follows formally from properties *i–v* of *Section 4*, just as in ref. 2 (sections 7.10 and 7.11). The basis $\{(b \diamond b')_{x,z} | (b, b') \in B_x^+ \times B_z^-\}$ is said to be *the canonical basis* of $V_x \otimes \Lambda_z$.

Section 5. For example, for any $b \in B_x^+$ and $b' \in B_z^-$, the elements $b\xi_x \otimes \eta_z$ and $\xi_x \otimes b'\eta_z$ belong to the canonical basis of $V_x \otimes \Lambda_z$, since they are fixed both by Θ and by $^-$.

Section 6. Consider the example of type A_1, with $I = \{i\}$. We set $x(i) = a \in \mathbf{N}$, $z(i) = c \in \mathbf{N}$. The canonical basis of $V_x \otimes \Lambda_z$ consists of the vectors

$$\varepsilon_{n,m} = \sum_{s \geq 0; s \leq n; s \leq m} v^{s(n-s-a)} \begin{bmatrix} c+s-m \\ s \end{bmatrix} E_i^{(n-s)}\xi_x \otimes F_i^{(m-s)}\eta_z$$

for various $n \in [0, a]$, $m \in [0, c]$ such that $n - m \leq a - c$, and of the vectors

$$\varepsilon_{n,m} = \sum_{s \geq 0; s \leq n; s \leq m} v^{s(m-s-c)} \begin{bmatrix} a+s-n \\ s \end{bmatrix} E_i^{(n-s)}\xi_x \otimes F_i^{(m-s)}\eta_z$$

for various $n \in [0, a]$, $m \in [0, c]$ such that $n - m \geq a - c$ (in the notation of ref. 2); the two definitions of $\varepsilon_{n,m}$ coincide when $n - m = a - c$.

For any k, the subspace of $V_x \otimes \Lambda_z$ spanned by the vectors $\varepsilon_{n,m}$ with $\min(n - a, m - c) \leq k$ is a U-submodule of $V_x \otimes \Lambda_z$. These subspaces form a composition series of the U-module $V_x \otimes \Lambda_z$ that is compatible with the canonical basis.

Section 7. The following three lemmas are proved using the definition of "crystal" lattices and bases at $v = \infty$ and their behavior under tensor product (see ref. 3, especially theorem 1 on p. 475). One first proves them with $\mathbf{Z}[v^{-1}]$ replaced by the ring of rational functions in v that are regular at ∞.

LEMMA 1. *Let* $x, t \in X^+$ *and let* $\gamma_{x,t}: V_{x+t} \to V_x \otimes V_t$ *be the unique* U*-linear map such that* $\gamma_{x,t}(\xi_{x+t}) = \xi_x \otimes \xi_t$. *Note that* $B_x^+ \subset B_{x+t}^+$.

(i) *For any* $b \in B_x^+$, *we have* $\gamma_{x,t}(b\xi_{x+t}) - b\xi_x \otimes \xi_t \in v^{-1}\mathcal{L}_x \otimes_{\mathbf{Z}[v^{-1}]} \mathcal{L}_t$.

(ii) *For any* $\beta \in \mathfrak{B}_{x+t}^+ - \mathfrak{B}_x^+$ *we have* $\gamma_{x,t}(\beta) - \beta_1 \otimes \beta_2 \in v^{-1}\mathcal{L}_x \otimes_{\mathbf{Z}[v^{-1}]} \mathcal{L}_t$ *for some* $\beta_1 \in \mathfrak{B}_x^+$ *and some* $\beta_2 \in \mathfrak{B}_t^+ - \{\xi_t\}$.

LEMMA 2. *Let* $t, z \in X^+$ *and let* $\gamma^{t,z}: \Lambda_{t+z} \to \Lambda_t \otimes \Lambda_z$ *be the unique* U*-linear map such that* $\gamma^{t,z}(\eta_{t+z}) = \eta_t \otimes \eta_z$. *Note that* $B_z^- \subset B_{t+z}^-$.

(i) *For any* $b \in B_z^-$, *we have* $\gamma^{t,z}(b\eta_{t+z}) - \eta_t \otimes b\eta_z \in v^{-1}\mathcal{L}_t' \otimes_{\mathbf{Z}[v^{-1}]} \mathcal{L}_z'$.

(ii) *For any* $\beta \in \mathfrak{B}_{t+z}^- - \mathfrak{B}_z^-$ *we have* $\gamma^{t,z}(\beta) - \beta_1 \otimes \beta_2 \in v^{-1}\mathcal{L}_t' \otimes_{\mathbf{Z}[v^{-1}]} \mathcal{L}_z'$ *for some* $\beta_1 \in \mathfrak{B}_t^- - \{\eta_t\}$ *and some* $\beta_2 \in \mathfrak{B}_z^-$.

LEMMA 3. *Let* $t \in X^+$ *and let* $\delta_t: V_t \otimes \Lambda^t \to \mathbf{Q}(v)$ *be the unique* U*-linear map such that* $\delta_t(\xi_t \otimes \eta_t) = 1$. [*Regard* $\mathbf{Q}(v)$ *as a* U*-module with* E_i, F_i *acting as* 0 *and* K_y *acting as* 1.] *If* $(\beta, \beta') \in \mathfrak{B}_t^+ \otimes \mathfrak{B}_t^-$ *is not equal to* (ξ_t, η_t), *then* $\delta_t(\beta \otimes \beta') \in v^{-1}\mathbf{Z}[v^{-1}]$.

Section 8. Given $x, t, z \in X^+$, we define a U-linear map $\tau = \tau_{x+t,t+z,x,z}: V_{x+t} \otimes \Lambda_{t+z} \to V_x \otimes \Lambda_z$ as the composition of $\gamma_{x,t} \otimes \gamma^{t,z}: V_{x+t} \otimes \Lambda_{t+z} \to V_x \otimes V_t \otimes \Lambda_t \otimes \Lambda_z$ with $1 \otimes \delta_t \otimes 1: V_x \otimes V_t \otimes \Lambda_t \otimes \Lambda_z \to V_x \otimes \Lambda_z$.

We have the following stability property.

THEOREM 2. (i) *If* $(b, b') \in B_x^+ \times B_z^-$, *then* $\tau((b \diamond b')_{x+t,t+z}) = (b \diamond b')_{x,z}$.

(ii) *If* $(b, b') \in B_{x+t}^+ \times B_{t+z}^- - B_x^+ \times B_z^-$, *then* $\tau((b \diamond b')_{x+t,t+z}) = 0$.

The map $V_{x+t} \otimes \Lambda_{t+z} \to V_{x+t} \otimes \Lambda_{t+z}$ defined like Ψ in *Section 4* (for $x + t$, $t + z$ instead of x, z) will be denoted by Ψ'. From the definitions we have that $\alpha_{x,z} = \tau\alpha_{x+t,t+z}$. Using property *i* of *Section 4* twice we deduce that $\Psi\tau\alpha_{x+t,t+z}(u) = \Psi\alpha_{x,z}(u) = \alpha_{x,z}(\bar{u})$ and $\tau\Psi'\alpha_{x+t,t+z}(u) = \tau\alpha_{x+t,t+z}(\bar{u}) = \alpha_{x,z}(\bar{u})$ for all $u \in \mathbf{U}$. Thus, $\Psi\tau\alpha_{x+t,t+z} = \tau\Psi'\alpha_{x+t,t+z}$; since $\alpha_{x+t,t+z}$ is surjective, it follows that

(iii) $\Psi\tau = \tau\Psi'$.

Under the assumptions of assertion *i*, the element $\tau((b \diamond b')_{x+t,t+z})$ belongs to $b\xi_x \otimes b'\eta_z + v^{-1}\mathcal{L}_{x,z}$ (see *Lemmas 1–3*) and is fixed by Ψ (see equation *iii*); by assertion *ii* of *Theorem 1*, it is equal to $(b \diamond b')_{x,z}$. Under the assumptions of assertion *ii*, the element $\tau((b \diamond b')_{x+t,t+z})$ belongs to $v^{-1}\mathcal{L}_{x,z}$ (see *Lemmas 1–3*) and is fixed by Ψ (see equation *iii*); hence it is zero, by assertion *i* of *Theorem 1*. This proves *Theorem 2*.

Section 9. In the U-module $V_x \otimes \Lambda_z$ we may consider the family of submodules consisting of the kernels of the surjective homomorphisms $\tau_{x,z,x',z'}$ for various x', z' in X^+ such that $x - x' = z - z' \in X^+$. From the previous theorem we see that the canonical basis of $V_x \otimes \Lambda_z$ is simultaneously compatible with all these subspaces (generalizing the example in *Section 6*). One can conjecture that (in the case where the Cartan matrix is positive definite), there exists a composition series of the U-module $V_x \otimes \Lambda_z$ all of whose members are compatible with the canonical basis.

Section 10. For any $\lambda \in X$ we denote

$$\mathbf{U}(\lambda) = \mathbf{U}/\left(\sum_{y \in Y} \mathbf{U}(K_y - v^{\lambda(y)})\right).$$

For any $x, z \in X^+$ such that $\lambda = z - x$, the linear map $\alpha_{x,z}: \mathbf{U} \to V_x \otimes \Lambda_z$ (see *Section 4*) factors through a linear map $\tilde{\alpha}_{x,z}: \mathbf{U}(\lambda) \to V_x \otimes \Lambda_z$. Let $I_{x,z} \subset \mathbf{U}(\lambda)$ be the kernel of $\tilde{\alpha}_{x,z}$. The next result follows easily from *Theorem 2*.

THEOREM 3. (i) *Given any* $(b, b') \in B^+ \times B^-$, *there is a unique element* $b \diamond_\lambda b' \in \mathbf{U}(\lambda)$ *such that* $\tilde{\alpha}_{x,z}(b \diamond_\lambda b') = (b \diamond b')_{x,z}$ *for any* x, z *in* X^+ *such that* $b \in B_x^+$, $b' \in B_z^-$, $z - x = \lambda$. *If either* $b \notin B_x^+$ *or* $b' \notin B_z^-$, *then* $\tilde{\alpha}_{x,z}(b \diamond_\lambda b') = 0$. *The element* $b \diamond_\lambda b'$ *is fixed by the map* $^-: \mathbf{U}(\lambda) \to \mathbf{U}(\lambda)$ *induced by* $^-: \mathbf{U} \to \mathbf{U}$.

(ii) *The elements* $b \diamond_\lambda b'$ *for* $(b, b') \in B^+ \times B^-$, *form a* $\mathbf{Q}(v)$*-basis of* $\mathbf{U}(\lambda)$. *This basis is simultaneously compatible with each of the subspaces* $I_{x,z}$ *defined in Section 10.*

Section 11. For any $p \in X$ we denote $\mathbf{U}_{[p]} = \{u \in \mathbf{U} | K_y u = v^{p(y)} u K_y \text{ for all } y \in Y\}$; thus $\mathbf{U} = \oplus_p \mathbf{U}_{[p]}$. Let $\pi_\lambda: \mathbf{U} \to \mathbf{U}(\lambda)$ be the natural projection. There is a natural structure of associative algebra (without 1) on $\tilde{\mathbf{U}} = \oplus_{\lambda \in X} \mathbf{U}(\lambda)$ inherited from that of $\mathbf{U}$. It is defined by the following requirement: for any $u \in \mathbf{U}_{[p]}$, $u' \in \mathbf{U}_{[p']}$ and any $\lambda, \lambda' \in X$, the product $\pi_\lambda(u)\pi_{\lambda'}(u')$ is equal to $\pi_{\lambda'}(uu')$ if $\lambda = \lambda' + p'$ and is zero otherwise. The elements $\pi_\lambda(1)$ form a set of orthogonal idempotents. Clearly, the elements $b \diamond_\lambda b'$ for $(b, b') \in B^+ \times B^-$ and $\lambda \in X$ form a $\mathbf{Q}(v)$-basis of $\tilde{\mathbf{U}}$; I call it the *canonical basis* of $\tilde{\mathbf{U}}$. It is easy to see that the A-submodule $\tilde{\mathbf{U}}_A$ spanned by these elements is an A-subalgebra (without 1) of $\tilde{\mathbf{U}}$.

Now $\tilde{\mathbf{U}}$ also inherits from $\mathbf{U}$ a structure close to a coalgebra structure; namely, for any three elements $\lambda, \lambda', \lambda'' \in X$ such that $\lambda = \lambda' + \lambda''$, there is a unique $\mathbf{Q}(v)$-linear map $\Delta_{\lambda',\lambda''}: \mathbf{U}(\lambda) \to \mathbf{U}(\lambda') \otimes \mathbf{U}(\lambda'')$ such that for any $u \in \mathbf{U}$ with $\Delta(u) = \Sigma u' \otimes u''$, we have $\Delta_{\lambda',\lambda''}(\pi_\lambda(u)) = \Sigma \pi_{\lambda'}(u') \otimes \pi_{\lambda''}(u'')$.

The structure constants (with respect to the canonical basis) of this "comultiplication", as well as those of the multiplication, belong to A.

One may conjecture that these structure constants are in fact in $\mathbf{N}[v, v^{-1}]$, generalizing the positivity property for comultiplication and multiplication proved in ref. 4.

Section 12. In the remainder of this paper I assume that the Cartan matrix is positive definite. Let $\mathbf{U}^* = \mathrm{Hom}(\mathbf{U}, \mathbf{Q}(v))$ be the dual space of $\mathbf{U}$; we can regard $\mathbf{U}^*$ as an associative algebra with multiplication $f, f' \mapsto ff'$, where $(ff')(u) = (f \otimes f')(\Delta(u))$ for all $u \in \mathbf{U}$.

Given $x, z \in X^+$, the surjective map $\alpha_{x,z}: \mathbf{U} \to V_x \otimes \Lambda_z$ (see *Section 4*) induces by passage to dual an injective $\mathbf{Q}(v)$-linear map $\alpha'_{x,z}: (V_x \otimes \Lambda_z)^* \to \mathbf{U}^*$. Let $\mathbf{U}^*(x, z)$ be its image. We have $\mathbf{U}^*(x, z) \subset \mathbf{U}^*(x + t, t + z)$ for any $t \in X^+$. One can check that under the multiplication in $\mathbf{U}^*$, we have $\mathbf{U}^*(x, z)\mathbf{U}^*(x', z') \subset \mathbf{U}^*(x + x', z + z')$. It follows that $\mathbf{A} = \Sigma_{x,z}\mathbf{U}^*(x, z)$ is a subalgebra (with 1) of $\mathbf{U}^*$, the *quantum coordinate algebra*.

Let $(b, b') \in B^+ \times B^-$ and let $\lambda \in X$. We can choose x, z in X^+ such that $b \in B_x^+$, $b' \in B_z^-$, $z - x = \lambda$. Let $g(b, b', x, z)$ be the linear form on $V_x \otimes \Lambda_z$ that takes value 1 at $(b \diamond b')_{x,z}$ and takes the value 0 on all other elements of the canonical basis. Then $\alpha'_{x,z}(g(b, b', x, z)) \in \mathbf{A}$ is independent of the choice of x, z by the stability property (*Theorem 2*); we denote it $b \diamond^\lambda b'$. The following result is easily verified.

THEOREM 4. *The elements* $\mathrm{b} \diamond^\lambda \mathrm{b}'$ *for various* $(\mathrm{b}, \mathrm{b}') \in \mathrm{B}^+ \times \mathrm{B}^-$ *and* $\lambda \in \mathrm{X}$ *form a* $\mathbf{Q}(\mathrm{v})$*-basis of* $\mathbf{A}$. *This basis is simultaneously compatible with each of the subspaces* $\mathbf{U}^*(\mathrm{x}, \mathrm{z})$.

There is a unique bilinear pairing $\langle , \rangle: \mathbf{A} \times \tilde{\mathbf{U}} \to \mathbf{Q}(v)$ such that the following holds: if $f \in \mathbf{A}$ and $\lambda \in X$ satisfy $f(uK_y) = v^{\lambda(y)}f(u)$ for all $y \in Y$, then, for any $u' \in \mathbf{U}$ and any $\mu \in X$, the value of $\langle f, \pi_\mu(u') \rangle$ is equal to $f(u')$ for $\mu = \lambda$ and to zero, for $\mu \neq \lambda$.

It is easy to check that $\langle b \diamond^\lambda b', b_1 \diamond_{\lambda_1} b_1' \rangle$ is equal to 1 if $b = b_1$, $b' = b_1'$, $\lambda = \lambda_1$ and it is zero, otherwise.

Section 13: Connections with earlier work. The idea to define the coordinate algebra (for $v = 1$) in terms of the enveloping algebra goes back to ref. 6. The algebra $\tilde{\mathbf{U}}$ appeared (for type A) in ref. 7, where a basis for it (presumably the same as the one in assertion *ii* of *Theorem 3*) was constructed by a quite different method; another approach to this basis (for $v = 1$) was later found in ref. 8. A definition (different from the one in *Section 12*) of the quantum coordinate algebra $\mathbf{A}$ together with a basis of it was given in ref. 9. Note that the approach in ref. 9 does not yield the compatibility of the basis with the various subspaces of $\mathbf{A}$ and does not yield a basis of $\tilde{\mathbf{U}}$. The possibility of describing the coordinate algebra using tensor products, as in *Section 12*, has been one of the ingredients in ref. 10; the maps $\tau_{x+t,t+z,x,z}$ (see *Section 8*) appeared in no. 18 of ref. 10 in a closely related context.

This work was supported in part by National Science Foundation Grant DMS 8702842.

1. Drinfeld, V. G. (1987) in Proceedings of the International Congress of Mathematicians, Berkeley 1986 (Am. Math. Soc., Providence, RI), pp. 798–820.
2. Lusztig, G. (1990) *J. Am. Math. Soc.* **3**, 447–498.
3. Kashiwara, M. (1991) *Duke Math. J.* **63**, 465–516.
4. Lusztig, G. (1991) *J. Am. Math. Soc.* **4**, 365–421.
5. Tanisaki, T. (1991) *Killing Forms, Harish-Chandra Isomorphisms, and Universal R-Matrices for Quantum Algebras*, preprint.
6. Kostant, B. (1966) *Proc. Symp. Pure Math.* **9**, 90–98.
7. Beilinson, A. A., Lusztig, G. & MacPherson, R. (1990) *Duke Math. J.* **61**, 655–677.
8. Ginzburg, V. (1991) *C. R. Acad. Sci. Paris* Ser. I **312**, 907–912.
9. Kashiwara, M. (1991) *Global Crystal Bases of Quantum Groups*, preprint.
10. Kazhdan, D. & Lusztig, G. (1991) *Int. Math. Res. Notes* **2**, 21–29.